Finite simple groups as automorphism groups of non-associative algebras and colored graphs

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Classification theorem

Finite simple groups

Every finite simple group is isomorphic to one of the following groups:

1. the cyclic groups of prime order,
2. the alternating groups of degree at least 5,
3. the groups of Lie type and Tits group,
4. one of 26 groups called the "sporadic groups".

Sporadic groups

Subgroups of the Monster Group

M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, HS, J_2, Co_1, Co_2, Co_3, McL, Suz,He, HN, Th, Fi_{22}, Fi_{23}, Fi'_{24}, B, M.

Pariahs

J_1, $O'N$, J_3, Ru, J_4, Ly.
These groups are impoverished by the fact that they are the only ones who are not included as subgroups in the Monster. The nature of these groups is different.

Fusion rules

Let $\mathbb{F}$ be a field and let $\mathcal{F} \subseteq \mathbb{F}$. A fusion table on $\mathcal{F}$ is a binary operation $* : \mathcal{F} \times \mathcal{F} \to 2^{\mathcal{F}}$.
By abuse of notation, we denote the fusion table by the same symbol $\mathcal{F}$, as the underlying set. We also talk about the fusion rules $\mathcal{F}$ meaning individual entries of $\mathcal{F}$.

Example

$$\mathcal{J}(\eta) = \begin{array}{|c||c|c|c|}
\hline
* & 0 & 1 & \eta \\
\hline
\hline
0 & 0 & \emptyset & \eta \\
\hline
1 & \emptyset & 1 & \eta \\
\hline
\eta & \eta & \eta & \{1,0\} \\
\hline
\end{array}$$

Notation: adjoint action

Let A be a (non-associative) algebra over a field $\mathbb{F}$.

- For $a \in A$, the (left) adjoint action $ad_a : A \to A$ is given by $ad_a(b) = ab$ for all $b \in A$.
- For $\lambda \in \mathbb{F}$, set $A_\lambda(a) = \{b \in A | ad_a(b) = \lambda b\} = \{b \in A | ab = \lambda b\}$.
- For $\Lambda \subseteq \mathbb{F}$, set $A_\Lambda(a) = \bigoplus_{\lambda \in \Lambda} A_\lambda(a)$.

Axis

Let $\mathcal{F} \subseteq \mathbb{F}$ be fusion rules.

Definition

An idempotent $a \in A$ is an $(\mathcal{F})$-axis if

- $A = A_{\mathcal{F}}(a)$; and
- $A_\lambda(a)A_\mu(a) \subseteq A_{\lambda*\mu}(a)$ for all $\lambda, \mu \in \mathcal{F}$.

This assumes that $1 \in \mathcal{F}$, as $ad_a(a) = aa = a = 1a$.
The axis a is primitive if $A_1(a) = \langle a \rangle$.

Axial algebra

Definition(J. Hall, F. Rehren, S. Spectorov (2015))

A commutative algebra A is a (primitive) ($\mathcal{F}$-)axial algebra if it is generated by (primitive) ($\mathcal{F}$-)axes.

Example

Take $\mathbb{F} = \mathbb{R}$ и $\mathcal{M} = \{0, 1, 1/4, 1/32\}$.

$*$	0	1	$1/4$	$1/32$
0	0	$\emptyset$	$1/4$	$1/32$
1	$\emptyset$	1	$1/4$	$1/32$
$1/4$	$1/4$	$1/4$	$\{1, 0\}$	$1/32$
$1/32$	$1/32$	$1/32$	$1/32$	$\{1, 0, 1/4\}$

A Majorana algebra of Ivanov is a primitive $\mathbf{M}$-axial algebra over $\mathbb{F} = \mathbb{R}$ satisfying some additional axioms. The 196,884-dimensional Griess algebra is a Majorana algebra, as are also many its subalgebras.

Algebras of Jordan type

An algebra of Jordan type η is a primitive $\mathcal{J}(\eta)$-axial algebra.
Jordan algebras generated by primitive idempotents are algebras of Jordan type $1/2$.

Group of n-transpositions

Let (G, C) be a group of n-transpositions:

- C is a normal set of involutions in G;
- $G = \langle C \rangle$;
- $|xy| \leq n$ for $x, y \in C$.

Sporadic groups of 3-transpositions

$Fi_{22}, Fi_{23}, Fi'_{24}$.

Sporadic groups of 6-transpositions

B, M.

Matsuo Algebras

The Matsuo algebra $M(G)$ (over $\mathbb{F}$ of characteristic not 2) has C as its basis.

$$ef = \begin{cases} e, \text{if } e = f; \\ 0, \text{if } ord(ef) = 2; \\ \dfrac{\eta}{2}(e + f - e^f), \text{if } ord(ef) = 3; \end{cases}$$

J. Hall, F. Rehren, S. Spectorov (2015)

The Matsuo algebra $M_\eta(G, D)$ is a primitive axial Jordan type algebra.

Miyamoto involution

Let $\mathcal{F}$ be a fusion rules.

Definition

$\mathcal{F}$ is C_2–grading if $\mathcal{F} = \mathcal{F}_+ \cup \mathcal{F}_-$ and
$\mathcal{F}_+ * \mathcal{F}_+ \subseteq \mathcal{F}_+, \mathcal{F}_- * \mathcal{F}_- \subseteq \mathcal{F}_+, \mathcal{F}_+ * \mathcal{F}_- \subseteq \mathcal{F}_-$.

$A_+(a) = \oplus_{\alpha \in \mathcal{F}_+} A_\alpha(a)$
$A_-(a) = \oplus_{\alpha \in \mathcal{F}_-} A_\alpha(a)$

Definition

Let a be an $\mathcal{F}$ axes. Put $\tau_a : A \to A$ such that $\tau_a A_+ = A_+$ and $\tau_a A_- = -A_-$ is automorphism (Miyamoto involution) of A.

Definition

Let D is a generators set of A. $May(A, D) = \langle \tau_d | d \in D \rangle$ is Miyamoto group.

Miyamoto groups

J. Hall, F. Rehren, S. Shpectorov (2015)

If $A = M_\eta(G, D)$, then $May(A, D) \simeq G/Z(G)$.

Axial algebras were constructed from simple 3-transposition groups.

Shpectorov and co-authors constructed axial algebras of Monster type for many sporadic groups.

Zelmanov question

E. Zelmanov

Construct C_2-graded axial algebras whose Miyamota group is isomorphic to a pariah sporadic simple group.

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I.G.

An axial algebra A with a C_2-graded fusion law is constructed. Dimension of A is 1463 and $Miy(A, D) \simeq J_2$.

Involutions graphs

Involutions scheme (I.G.2025)

Let G be a group and Ω a normal set of involutions of G. We define an equivalence relation on the sets $\Omega \times \Omega$: $(a, b) \sim (c, d)$ if $|ab| = |dc|$. The set of involutions Ω with such a partition is the involutions scheme and is denoted by the same letter Ω.

When a group acts doubly transitively on a normal set of involutions Ω, then the involutions scheme turns out to be an associative scheme (there are intersection coefficients).

Involutions graphs (I.G.2025)

Let Ω be a normal set of involutions. A complete colored graph on the set Ω constructed according to the involution scheme on Ω is called an involution graph.

Automorphism group

Definition

The permutation ϕ on the set Ω is called an automorphism (strong) of the involution scheme Ω if from the fact that $(a, b) \sim (c, d)$ it follows that $(\phi(a), \phi(b)) \sim (\phi(c), \phi(d))$ $\big((a, b) \sim (\phi(a), \phi(b))\big)$.

Questions

Let G be a simple group, Ω be the class of conjugate involutions in G.

Question

Is it true that the automorphism group is isomorphic to the strong automorphism group of the involution scheme Ω?

It is clear that $Aut(G) \leq Aut(\Omega)$.

Question

Is it true that $Aut(\Omega) \simeq Aut(G)$?

Let Ω_i be a subgraph of the involution graph Ω with edges corresponding to color i. It is clear that $SAut(\Omega) \simeq \bigcap Aut(\Omega_i)$. Thus, if for some i $Aut(\Omega_i) = Aut(G)$ is true, then $SAut(\Omega) = Aut(G)$.

I.G. 2025

Let G be a sporadic simple group distinct from M and BM, and Ω be an arbitrary class of involutions of G. Then $Aut(\Omega) \simeq Aut(G)$.

Definitions

Faithful odd transposition group

A pair (G, T) is called a faithful odd transposition group if G is a periodic group and T is a normal subset of involutions of G such that $\langle T \rangle = G$, and $|ab|$ for any $a, b \in T$ is odd.

$GM(n)$-type group

A faithful odd transposition group (G, T) is called generalized Moufang type n, or $GM(n)$-type for short, if order of the product of any two distinct elements of T is n.

Odd transposition algebra

Let (G,T) be a faithful odd transposition group, $\Omega_T = \{|ab| | a,b \in T\} \setminus \{1\}$. For each number $n \in \Omega_T$ we define an element $\eta_n \in \mathbb{F} \setminus \{0,1\}$. Let $\Gamma_T = \{\eta_n | n \in \Omega_T\}$. We construct an $\mathbb{F}$-algebra with basis T as follows:

$$a * b = \begin{cases} a, & \text{if } b = a, \\ b^a + \delta(b^a), & \text{where } \delta(b^a) = \eta_n \sum_{x \in I(a,b) \setminus \{b^a\}} x \text{ and } n = |ab|. \end{cases}$$

We denote such an algebra by $A_{\mathbb{F}}(G,T,\Gamma_T)$ and call it the odd transposition algebra. In the case when $|\Omega_T| = 1$ i.e. (G,T) is a $GM(n)$-group, we write $A_{\mathbb{F}}(G,T,\eta)$.

Properties

I.G.

Let (G,T) be a $GM(p)$-type group and $A = A_{\mathbb{F}}(G,T,\eta)$, where $\eta \in \mathbb{F}$ be such that $1, \eta(p-2)+1, 1-\eta$ and $\eta - 1$ are distinct elements of $\mathbb{F}$. The following assertions hold:

1. All elements of T are primitive left semisimple idempotents in A.
2. If $\mathbb{F}$ contains the $\frac{p-1}{2}$ roots of -1, then any element $a \in T$ is a right semisimple idempotent in A.
3. A contains no right ideals.
4. For each idempotent $a \in T$, the decomposition $A = A_a^+ \oplus A_a^-$ defines a C_2-grading of A. With respect to these gradings $Miy(A,T) \simeq G/Z(G)$.

I.G.

Let (G, T) be a two generated group of $GM(5)$-type, $\eta = -\frac{1}{3}$ and $\mathbb{F}$ is a field of good characteristic relative to $(5, \eta)$. Every idempotent in T is a left-axis of $\mathcal{M}(4/3, -4/3)$-type.

I.G.

Let (G, T) be a group of $GM(p)$-type, where $p > 5$, $\mathbb{F}$ is a field, $\eta \in \mathbb{F}$, and characteristic of $\mathbb{F}$ is good related to (p, η) and $A = A_{\mathbb{F}}(G, T, \eta)$. All idempotents of T are not a left $\mathcal{M}(\alpha, -\alpha)$-axis for any value of the parameter η.

Burnside's conjecture

Let $G = B(n,p).\langle t\rangle$, where p is a prime greater than 2 and t is an automorphism of order 2 of $B(n,p)$ that inverts free generators, $T = t^G$, $\mathbb{F}$ be a field of good characteristics relative to (p,η). Then $Miy(A_{\mathbb{F}}(G,T,\eta),T) \simeq G/Z(G)$, where $Z(G)$ is the center of G and G is finite if and only if $A_{\mathbb{F}}(G,T,\eta)$ is finite-dimensional.

Definition

We say that an $\mathbb{F}$-algebra $A(T)$ with basis T is an algebra of $GM(p, \eta)$-type if the following statements hold:

1. every element of T is idempotent;
2. for any pair of elements $a, b \in T$, $\langle\langle a, b\rangle\rangle \simeq A_{\mathbb{F}}(\langle a, b\rangle, I(a, b), \eta)$ is a dihedral algebra of dimension p;
3. for any element $a \in T$, there exists a subset $\Omega_a = \{t_1, ...\} \subseteq T$ such that $T = \{a\} \bigcup_{t \in \Omega_a} (I(a, t) \setminus \{a\})$;
4. for any $a \in T$, the linear mapping $\phi_a : A \to A$ defined by the rule $\phi_a(b) = b^a$ for any $b \in T$ is an automorphism of A.

I.G.

Let $A(T)$ be an algebra of $GM(p,\eta)$-type. Then $M = Miy(A,T)$ is a $GM(p)$-group and $A(T) \cong A_{\mathbb{F}}(M,T,\eta)$.

Thank you!